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Symmetry properties of exact energy solutions to the Harper equation and related q-normalizations

E Papp and C Micu

Department of Physics, Str. Victoriei 76, North University of Baia Mare, RO-4800 Baia Mare, Romania

E-mail: epapp@arad.ro

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Abstract. Proofs are given that exact solutions to the wavefunctions of the Harper equation can be established in an explicit manner by resorting to three-term recurrence relations implied by a q-calculus approach proposed previously. The q-normalization of wavefunctions resulting in the appearance of peaks is discussed. The exact $Q = 6$ energy solution has also been derived and analysed in some more detail. There are reasons to say that fractal structures concerning band and gap distributions have to be accounted for if $Q \geqslant 6$.

1. Introduction

The discrete second-order Harper equation [1–4] is a long standing problem [5–9], which has received renewed interest in several fields such as level statistics in quantum systems with unbounded diffusion [10, 11], ultracold atoms in bichromatic light waves [12], mesoscopic rings threaded by a magnetic flux [13], microwave realizations [14] and last but not least the fractional quantum Hall effect [15]. Challenging mathematical problems such as the hierarchical multifractal structure of the Harper spectrum [7, 16], relationships between self-similarity, quasiperiodicity and localization [17–19], links between hierarchical multifractality and semi-Poisson bandwidth distributions [20], or the derivation of generalized fractal dimensions [21], have also been investigated. On the other hand, implicit Bethe ansatz-like solutions have been presented by starting from a symmetrized q-difference form of this equation [22]. It is understood that this equation refers to the middle point of the Brillouin-zone, i.e. to $k_1 a = k_2 b = \pi/2$. The components of the wavevector are k_i $(i = 1, 2)$, whereas a and b are the lattice spacings, as usual. The q-parameter alluded to above has the form $q = \exp(\mathrm{i}\,\pi\beta)$, where $\beta = \Phi/\Phi_0$ is a commensurability parameter expressing the number of flux quanta per unit cell. Here we are concerned with rational β-values like $\beta = P/Q$, where P and Q are mutually prime integers. Accordingly, $q^{2Q} = 1$, so that the Q-parameter is responsible for the dimension of the related $SL_q(2)$ representation [22]. This results in multiplets for which $Q = 1, 2, 3, \ldots$, respectively. However, so far explicit solutions have been discussed in some detail for the zero energy only [23]. This motivates us to look for the derivation of further explicit solutions, their symmetry properties included.

In this context the first five energy multiplets ($Q = 1, 2, \ldots, 5$) to the symmetrized q-difference Harper equation mentioned above

$$\mathcal{H}\psi(z) = \mathrm{i}\left(\frac{1}{z} + qz\right)\psi(qz) - \mathrm{i}\left(\frac{1}{z} + \frac{z}{q}\right)\psi(q^{-1}z) = E\psi(z) \tag{1}$$

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have been established in an explicit manner by resorting to the q-calculus [24]. This amounts to apply the Jackson derivative [25, 26], which has also been reintroduced recently in connection with the radial reduction of the covariant $SO_q(N)$ derivative [27, 28]. Besides deriving the exact $Q = 6$ energy solution, we shall then use this opportunity to present additional comments concerning the symmetry attributes of energy multiplets. Typical patterns exhibited in the field dependence of the $Q = 6$ solution will be discussed in some more detail. We shall also derive explicit results concerning the wavefunctions, with a special emphasis on pertinent q-normalizations.

2. Preliminaries and notation

Using the symmetrized Jackson derivative

$$\partial_z^{(q)} f(z) = \frac{\mathrm{d}_q f(z)}{\mathrm{d}_q z} = \frac{f(qz) - f(q^{-1}z)}{z(q - q^{-1})} \tag{2}$$

enables us to rewrite (1) equivalently as

$$\left(\partial_z^{(q)} + z\partial_z^{(q)} z\right) \psi(z) = W\psi(z) \tag{3}$$

where

$$W = -\mathrm{i}\frac{qE}{q^2 - 1}. \tag{4}$$

Accordingly, $E = E^*$ if $W = W^*$, where the star superscript denotes complex conjugation. It is also clear that $E = 0$ if $q^2 = 1$, whereas $E = \mp 2W$ if $q = \pm\mathrm{i}$. So the $q^2 = 1$ choice represents a special limiting case for which $E = 0$ irrespective of W, so that it can be ruled out in the following. Polynomial solutions like

$$\psi_Q(z) = \sum_{n=0}^{Q-1} c_n z^n \tag{5}$$

where $c_0 = 1$, can then be established by virtue of the three-term recurrence relation

$$[n+1]_q c_{n+1} + [n]_q c_{n-1} = W c_n \tag{6}$$

where

$$[n]_q \equiv q^{1-n}[[n]]_{q^2} = \frac{q^n - q^{-n}}{q - q^{-1}}. \tag{7}$$

Accordingly, the energies are established by virtue of the algebraic equation

$$c_Q \equiv c_Q(q, W) = \frac{U_Q(q, W)}{[[Q]]_{q^2}!} \neq 0. \tag{8}$$

The numerator exhibits the general form

$$U_Q = U_Q(q, W) = q^{\omega_Q} P_{\gamma_Q}^{(Q)}(q^2, W) \neq 0 \tag{9}$$

which comes from a reasonable generalization of concrete results derived before [24]. One has

$$\gamma_Q = \begin{cases} Q(Q-2)/2 & Q = \text{even} \\ (Q-1)^2/2 & Q = \text{odd} \end{cases} \tag{10}$$

where $\omega_Q = I(Q/2)$ denotes the largest integer part of $Q/2$. We then have to solve in terms of W the algebraic equation

$$P_{\gamma_Q}^{(Q)}(q^2, W) = 0 \tag{11}$$

where $P_{\gamma_Q}^{(Q)}$ is a polynomial of degree Q and γ_Q in W and q^2, respectively. This yields a number of Q ordered real energy solutions like

$$E_l^{(Q)}(q) = E_l^{(Q)^*}(q) = -E_l^{(Q)}(1/q) = \frac{\mathrm{i}}{q}(q^2-1)q^{2-Q}\widetilde{w}_l^{(Q)}(q^2) \tag{12}$$

increasing with l, where $l = 1, 2, \ldots, Q$ and $q^* = 1/q$, and where the star superscript denotes complex conjugation.

After having arrived at this stage, some further remarks are in order. First, it should be noted that the $\widetilde{w}_l^{(Q)}(q^2)$ factor is expressed by continued square-root expressions with a complexity increasing with Q. We should also note that the energies established in this manner exhibit the symmetry property

$$E_l^{(Q)}(q) = -E_{Q-l+1}^{(Q)}(q). \tag{13}$$

This shows that the energy values are centred around $E_{(Q+1)/2}^{(Q)}(q) = 0$ if Q is an odd number. In addition, (13) indicates that $-E$ is an energy solution if E is also an energy solution, and conversely. The energy reflection-symmetry implied in this manner can be understood in terms of the transfer-matrix approach [4, 29] or by invoking the underlying $SL_q(2)$ symmetry [30]. One notes that the energies presented above agree with those defined by virtue of the Bethe ansatz (BA) method. Choosing, for example, $Q = 2$ and using (5) and (6) presented in [22], one finds $z_1^{BA} = \pm \mathrm{i}q$ and $E_{BA}^{(2)} = \mp 2$. Then $W_{BA}^{(2)} = \pm 1$, which reproduces identically (25) in [24].

3. The derivation of the $Q = 6$ energy solution

Next we have to say that closed formulae become both lengthy and hardly derivable if $Q \geqslant 6$, so that we have to resort, in general, to numerical evaluations. So one has

$$\begin{aligned} P_{12}^{(6)}(q^2, W) = {} & q^{12}W^6 - \left(q^{20} + 3q^{18} + 6q^{16} + 10q^{14} + 15q^{12} + 10q^{10} + 6q^8 + 3q^6 + q^4\right) W^4 \\ & + \left(q^{24} + 5q^{22} + 16q^{20} + 33q^{18} + 53q^{16} + 71q^{14} + 81q^{12} + 71q^{10} + 53q^8\right. \\ & \left. + 33q^6 + 16q^4 + 5q^2 + 1\right) W^2 - \left(q^{24} + 4q^{22} + 10q^{20} + 18q^{18} + 27q^{16}\right. \\ & \left. + 34q^{14} + 37q^{12} + 34q^{10} + 27q^8 + 18q^6 + 10q^4 + 4q^2 + 1\right) \end{aligned} \tag{14}$$

for $Q = 6$. Of course (14) is a direct byproduct of recurrence relations, which also means that the $q^{2Q} = 1$ fixing has not yet been carried out. First, we have to state that the numerical results produced by (11) and (14) are $W_1^{(6)} = -W_6^{(6)} = -3.086\,130$, $W_2^{(6)} = -W_5^{(6)} = -1.514\,136$ and $W_3^{(6)} = -W_4^{(6)} = -0.428\,006$ for typical q-values like $q = \exp(\mathrm{i}(2n+1)\pi/6)$, where $n = 0, 2, 3, 5, \ldots$. However, equation (14) is exactly solvable by applying standard formulae for cubic equations. This yields six (now non-ordered) W-roots like

$$\widetilde{W}_{\pm}^{(1)}(q) = \pm\left(F_+^{1/3} + F_-^{1/3}\right) \tag{15}$$

$$\widetilde{W}_{\pm}^{(2)}(q) = \pm\left(\varepsilon_+ F_+^{1/3} + \varepsilon_- F_-^{1/3}\right) \tag{16}$$

and

$$\widetilde{W}_{\pm}^{(3)}(q) = \pm\left(\varepsilon_- F_+^{1/3} + \varepsilon_+ F_-^{1/3}\right) \tag{17}$$

Table 1. The coefficients b_n and c_n characterizing the S and D functions.

n	b_n	c_n	c_{n+13}
0	844/3	9 363 772	86 850
1	1604/3	18 179 350	32 825
2	4136/9	16 628 084	10 896
3	9704/27	14 327 772	3 093
4	772/3	11 625 408	722
5	505/3	8 876 342	130
6	896/9	6 370 892	16
7	473/9	4 291 796	1
8	73/3	2 707 724	0
9	86/9	1 595 100	0
10	3	873 785	0
11	2/3	442 612	0
12	2/27	205 742	0

where

$$\varepsilon_{\pm} = -\frac{1}{2} \pm \mathrm{i}\frac{\sqrt{3}}{2} \tag{18}$$

and

$$F_{\pm} = -\frac{S}{2} \pm \sqrt{D}. \tag{19}$$

Furthermore,

$$S = -\sum_{n=0}^{12} b_n \gamma_{2n} \tag{20}$$

and

$$D = -\frac{1}{108}\sum_{n=0}^{20} c_n \gamma_{2n} \tag{21}$$

where

$$\gamma_{2n} = q^{2n} + q^{-2n}. \tag{22}$$

The coefficients characterizing (20) and (21) are displayed in table 1, so that the $Q = 6$ energy solution is completely determined.

Using (4) and (15)–(17) produces the original energies as

$$E_{\pm}^{(i)} \equiv -\widetilde{E}_{\pm}^{(i)}(q) = -2\sin\frac{\pi P}{6}\widetilde{W}_{\pm}^{(i)}(q) \tag{23}$$

where now $q = \exp(\mathrm{i}\pi P/6)$ and $i = 1, 2, 3$. Next we note that useful information concerning typical patterns can be established by analysing the $x \equiv P$ dependence of $\widetilde{E}_{\pm}^{(i)}(q)$ such as given by (23). This amounts to considering the real and imaginary parts characterizing $\widetilde{E}_{\pm}^{(i)}(q)$. Note that, except for the zero energy, both Re $\widetilde{E}_{+}^{(1)}(q)$ and Im $\widetilde{E}_{+}^{(1)}(q)$ are characterized by an irregular fractal-like behaviour characterizing selected x points and/or x regions, as shown in figure 1. This concerns x points for which the quotient P/Q is reducible, i.e. for $x = 2, 3$ and 4, although for $x = 4$ the magnitude of Im $\widetilde{E}_{+}^{(1)}(q)$ is negligibly small. In addition, we remark that Im $\widetilde{E}_{+}^{(1)}(q)$ is zero in the complementary x regions. Such regions should then

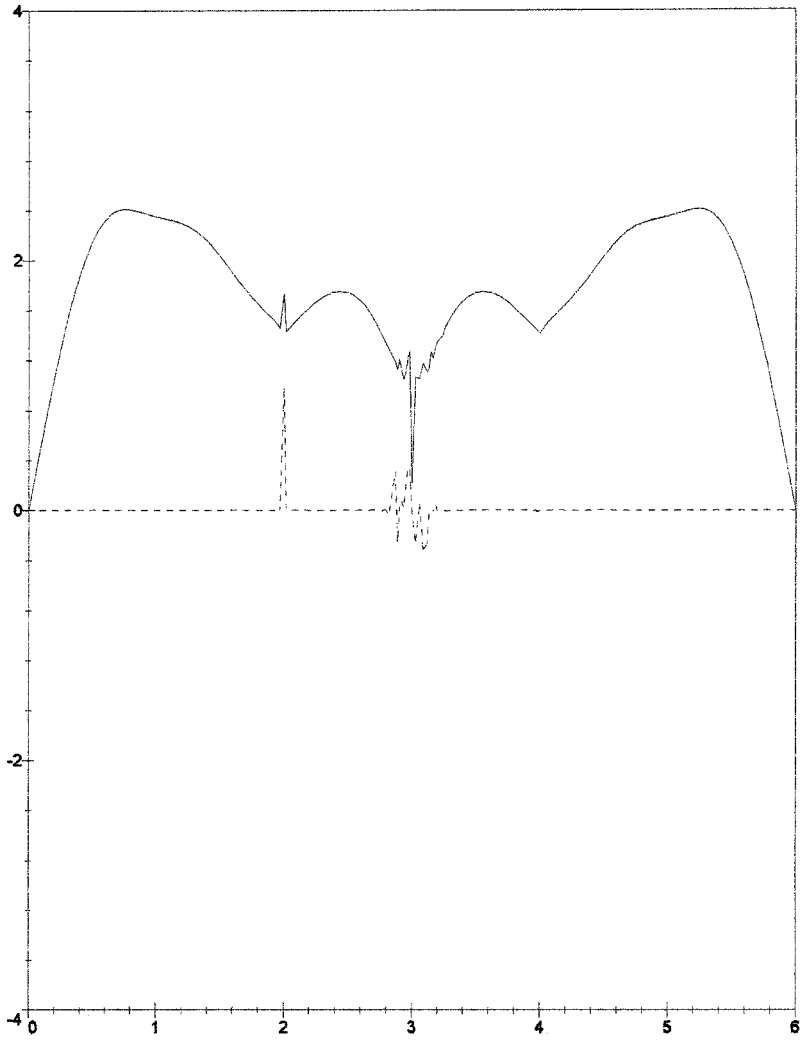

Figure 1. The $x \equiv P$ dependence of Re $\widetilde{E}_{+}^{(1)}(q)$ (full curve) and Im $\widetilde{E}_{+}^{(1)}(q)$ (broken curve) for $0 \leqslant x \leqslant 6$.

be viewed as being responsible for the pertinent band structure. Next, note the broad fractal structures centred around $x = 3$ in the x dependence of both Re $\widetilde{E}_{+}^{(1)}(q)$ and Im $\widetilde{E}_{+}^{(1)}(q)$. This indicates that within this latter region we have to account for the onset of an energy gap, this time exhibiting a larger width. In contradistinction, the gaps relying on $x = 2$ and 4 have a rather small and negligible width, respectively. Further sequences of competing band and gap structures are located dominantly within $2 \lesssim x \lesssim 4$ for $i = 2$ and 3, as shown in figures 2 and 3, respectively. Such sequences are enclosed within two large gaps. Indeed, both Im $\widetilde{E}_{+}^{(2)}(q)$ and Im $\widetilde{E}_{+}^{(3)}(q)$ exhibit fractal structures for $0 < x \lesssim 2$ and $4 \lesssim x < 6$, so that these latter regions are characterized by energy gaps only. It should also be mentioned that the x dependence of Re $\widetilde{E}_{+}^{(i)}(q)$ and Im $\widetilde{E}_{+}^{(i)}(q)$ is periodic with period $T = 2Q = 12$, which proceeds, of course, irrespective of $i = 1, 2$ and 3. We have to mention that competitions between band and gap structures have also been established in the case of one-dimensional

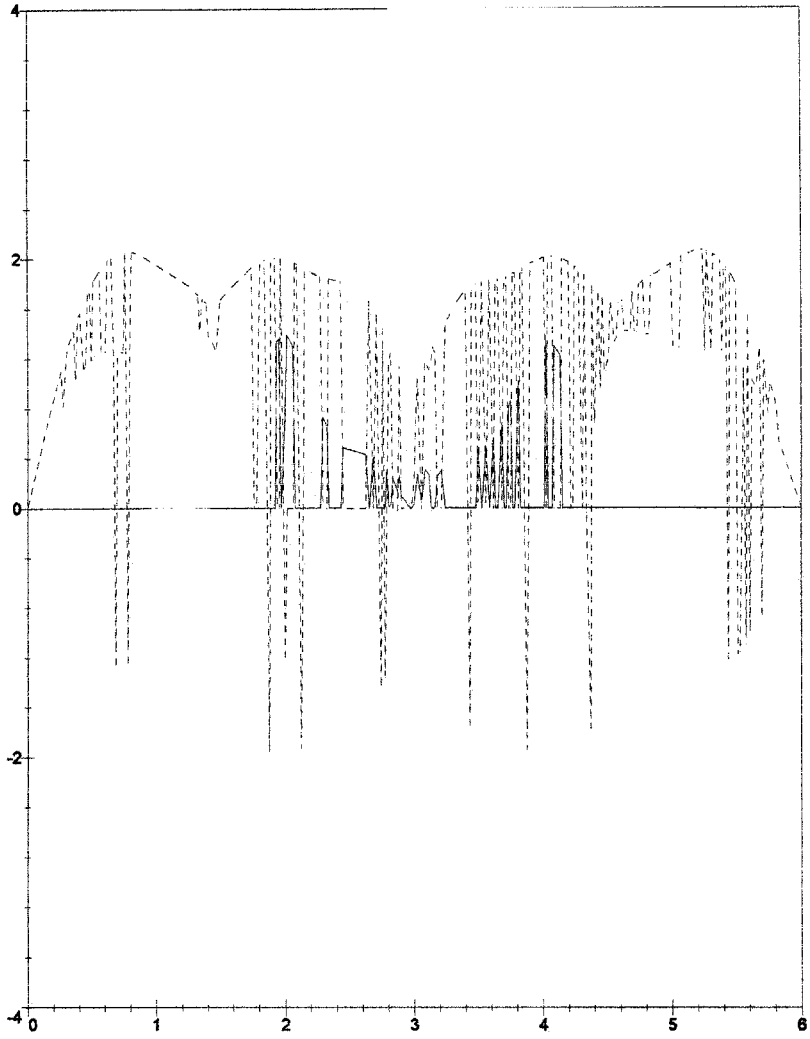

Figure 2. The $x \equiv P$ dependence of Re $\widetilde{E}_{+}^{(2)}(q)$ (full curve) and Im $\widetilde{E}_{+}^{(2)}(q)$ (broken curve) for $0 \leqslant x \leqslant 6$. Note a sequence of competing bands and gaps which is located approximately within $2 \lesssim x \lesssim 4$.

fermion systems with incommensuration, as shown in figure 1 in [29]. These latter results are able to support interpretations made above, as (1) represents a particular case of such systems. Moreover, performing a fractal analysis of band and gap structures displayed in figures 2 and 3 may result, at least tentatively, in the derivation of statistical distributions, like those presented before [20].

The special patterns displayed in figures 1–3 can be understood as fingerprints of the celebrated Hofstadter butterfly characterizing the Harper spectrum [6]. This differs, however, from the $x \equiv P$ dependence exhibited by the energies previously found for $Q = 1, 2, \ldots, 5$ [24], for which the imaginary parts are definitely zero.

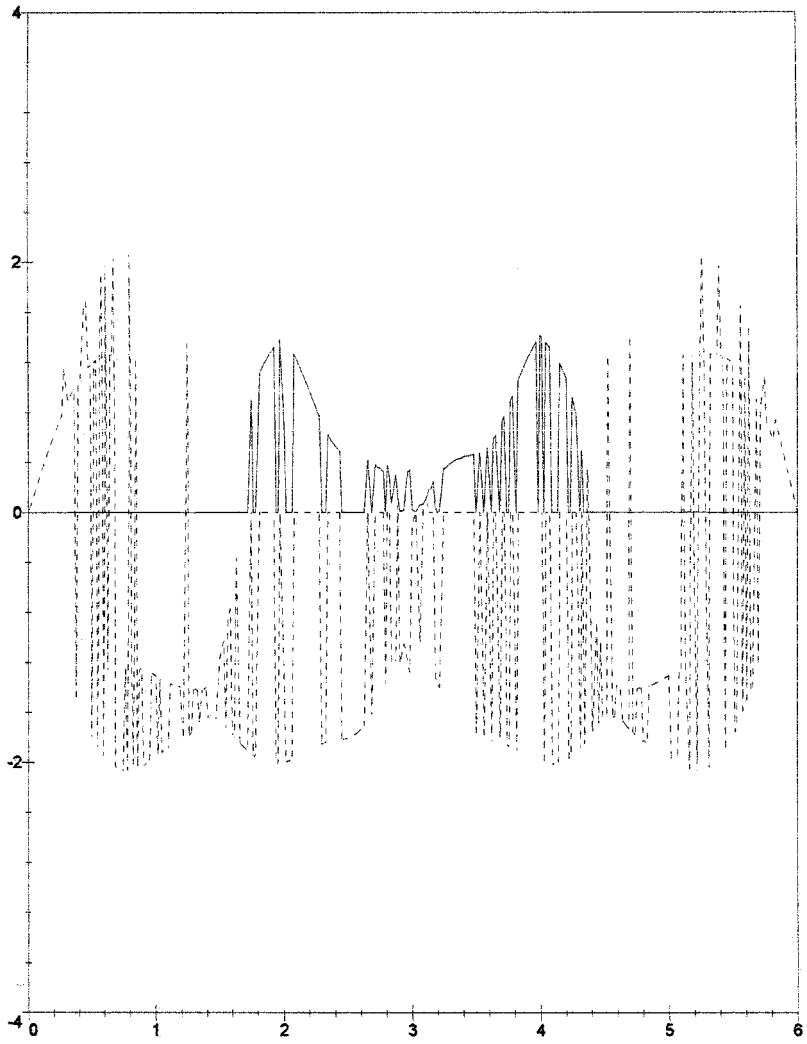

Figure 3. The $x \equiv P$ dependence of Re $\widetilde{E}_{+}^{(3)}(q)$ (full curve) and Im $\widetilde{E}_{+}^{(3)}(q)$ (broken curve) for $0 \leqslant x \leqslant 6$. A sequence of competing bands and gaps is located again within $2 \lesssim x \lesssim 4$.

4. The q-normalization of wavefunctions

The wavefunctions characterizing (1) are polynomials in z of degree $Q - 1$, in accord with the dimension of $SL_q(2)$ representations [22]. In addition, an explicit zero-energy solution has also been derived [31]. On the other hand, equation (3) can be viewed as a q-deformed wave equation, in which the pertinent deformation parameter is a root of unity exhibiting an inherent physical meaning. This shows, in turn, that corresponding wavefunctions have to be normalized definitely by resorting to the q-integral [26, 32]. For this purpose we have to keep in mind the fact that novel features are able to emerge by virtue of the more general significance of the quantum-group description [33]. Of course, there are also other solvable physical systems exhibiting inherent quantum-group symmetries, such as the $SU_q(2)$ symmetry characterizing, for example, the Heisenberg *XXZ* spin-chain [34]. Moreover, the non-commutative Euclidean

space is described by a corresponding deformation parameter, which leads to the onset of the radial q-derivative, too (see, e.g., [28]). Either way, the application of the q-calculus provides the proper treatment of q-deformed wave equations, which results in further generalizations and interesting rich structures.

We next perform the q-normalization needed by choosing, for convenience, the normalization interval $z \in [-1, 1]$, such that $z^* = z$. Accordingly, one deals with the q-integral [26, 32]

$$\int_{-1}^{1} f(z)\, \mathrm{d}_q z = \left(q - \frac{1}{q}\right) \sum_{j=0}^{\infty} \frac{1}{q^{2j+1}} \left(f\left(\frac{1}{q^{2j+1}}\right) + f\left(\frac{-1}{q^{2j+1}}\right)\right) \tag{24}$$

where $|q| > 1$, while a similar integral can also be established for $|q| < 1$. Accordingly,

$$\int_{-1}^{1} z^n\, \mathrm{d}_q z = \frac{1}{[n+1]_q} z^{n+1} \,|_{-1}^{1} \tag{25}$$

which works irrespective of q and which produces non-zero values for even n-exponents provided, of course, that $[n+1]_q \neq 0$. Conversely, one has

$$\frac{\mathrm{d}_q z^{n+1}}{\mathrm{d}_q z} = [n+1]_q z^n \tag{26}$$

which shows clearly that the above q-integral comes precisely from the inversion of the q-derivative (2). These results open the way to the consistent q-normalization of wavefunctions.

First we have to remark that applying (5) and (6) yields, up to normalization constants, the wavefunctions $\psi_1(q, z) = 1$,

$$\psi_2(q, z) = 1 + Wz \tag{27}$$

and

$$\psi_3(q, z) = \left(1 + Wz + \frac{q}{q^2+1}\left(W^2 - 1\right) z^2\right) \tag{28}$$

for $Q = 1, 2$ and 3. So $W_{l_1}^{(1)} = 0$, $W_{l_2}^{(2)} = \pm 1$ and

$$W_{l_3}^{(3)} = -\frac{1}{q}\sqrt{q^4 + 3q^2 + 1}, \quad 0, \quad \frac{1}{q}\sqrt{q^4 + 3q^2 + 1} \tag{29}$$

where now $l_2 = -1, 1$ and $l_3 = -1, 0, 1$. These energies yield the explicit wavefunctions

$$\psi_2^{(\pm)}(q, z) = 1 \pm z \tag{30}$$

$$\psi_3^{(0)}(q, z) = 1 - \frac{q}{q^2+1} z^2 \tag{31}$$

and

$$\psi_3^{(\pm)}(q, z) = \left(1 \pm \frac{1}{q}\sqrt{q^4 + 3q^2 + 1}\, z + \frac{q^2+1}{q} z^2\right) \tag{32}$$

which can be readily normalized by virtue of (25). We then find the amplitudes

$$I_2^{(\pm)}(q) = \frac{1}{2}\int_{-1}^{1} \left|\psi_2^{(\pm)}(z)\right|^2 \mathrm{d}_q z = \frac{\left(q^2+1\right)^2}{q^4 + q^2 + 1} \pm \frac{q\,(\exp(4\mathrm{i}\pi) - \exp(2\mathrm{i}\pi))}{q^2+1} \tag{33}$$

$$I_3^{(0)}(q) = \frac{1}{2}\int_{-1}^{1} \left|\psi_3^{(0)}(z)\right|^2 \mathrm{d}_q z$$
$$= 1 - \frac{2q^3}{(q^2+1)(q^4+q^2+1)} + \frac{q^6}{(q^2+1)^2(q^8+q^6+q^4+q^2+1)} \tag{34}$$

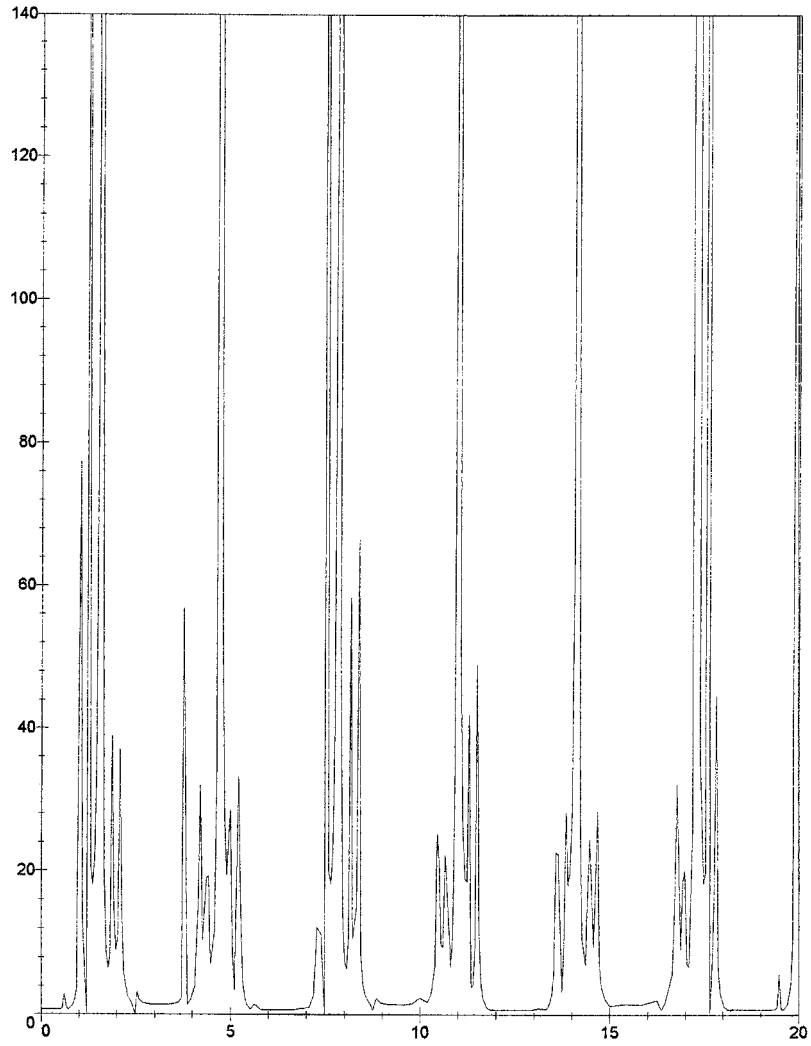

Figure 4. The ω dependence of $|I_3^{(0)}(q)|$. The present peaks looking like sextuplets rely on the P_s-pairs (1, 2), (4, 5), (7, 8), (10, 11), (13, 14) and (16, 17). The valleys correspond to $P_v = 3, 6, 9, 12, 15$ and 18. A further superposition begins to appear for $P_s = 19$.

and similarly for $\psi_3^{(\pm)}(q, z)$. The $\frac{1}{2}$-factors in the front of normalization integrals have been inserted just for convenience. The normalized wavefunctions are then given by

$$\varphi_Q^{(l_Q)}(q, z) = \left|2I_Q^{(l_Q)}(q)\right|^{-1/2} \psi_Q^{(l_Q)}(q, z). \tag{35}$$

It is also clear that the last term in the right-hand side of (33) is subject to regularization, as in this case $[2]_q = 0$. One would then obtain

$$\left|I_2^{(\pm)}\right| = |\pm 2\mathrm{i}| = 2 \tag{36}$$

which expresses the regularized version of (33) under the $q^2 = -1$ limit. Next, one has the symmetry property

$$\psi_Q^{(l_Q)}(q, z) = \psi_Q^{(l_Q)^*}(q, z) = \psi_Q^{(l_Q)}(1/q, z) \tag{37}$$

which proceeds in conjunction with (12). We remark that the wavefunctions corresponding to different W energies are not automatically orthogonal. Indeed, choosing, for example, $Q = 2$ and inserting $q^2 = -1$ one obtains

$$\int_{-1}^{1} \psi_2^{(1)^*}(q, z)\psi_2^{(-1)}(q, z)\, \mathrm{d}_q z = \frac{4}{2 + q^2} = 4 \tag{38}$$

where $W = \pm 1$. The understanding is that one has $E = \pm 2$ in both cases. Indeed, using the Harper Hamiltonian yields

$$\mathcal{H}(1 \pm z) = \frac{\mp 2\mathrm{i}}{q}(1 \pm z) \tag{39}$$

so that

$$\mathcal{H}^2(1 \pm z) = 4(1 \pm z) \tag{40}$$

where $q = \pm \mathrm{i}$. This shows that the non-orthogonality mentioned above is safely explained.

Inserting $q \equiv \exp(\mathrm{i}\omega)$ and considering, for example, that $\omega \in [0, 20]$, we found that the ω dependence of $I_Q^{(l_Q)}(q)$ is characterized by peaks. Furthermore, we have to realize that peaks implied in this way serve to reveal the very structure of the commensurability parameter $\beta = P/Q$. Choosing, for instance, $Q = 3$, one has admissible P-values like $P_s = 1, 2, 4, 5, 7, 8, \ldots$, which are not divisible by 3. In contradistinction, there are complementary P-values like $P_v = 3, 6, 9, 12, 15, \ldots$, which have to be ruled out. Correspondingly, the ω dependence of $\left|I_3^{(0)}(q)\right|$, where now $l_3 = 0$, is characterized by peaks looking like sextuplets, which are centred around P_s-pairs such as given by $(1, 2)$, $(4, 5)$, $(7, 8)$, $\ldots$, as illustrated in figure 4. The forbidden P_v-values are then responsible for smooth 'valleys' lying in between like $P_v = 3, 6, 9, \ldots$. We can then say that such peaks can be viewed as genuine manifestations of the multifractality of the Harper spectrum, now for non-integer but rational values of the commensurability parameter. Other cases can be treated in a similar manner.

5. Conclusions

In this paper the $Q = 6$ energy solution to the Harper equation has been derived and discussed. The $x \equiv P$ dependence of the energies established in this manner is characterized by special patterns indicating the existence of fractal-like energy-band and energy-gap distributions, as shown in figures 1–3. Such gaps rely on x regions in which the imaginary part of the energy is non-zero. This provides useful insights towards a deeper understanding of the multifractal attributes of the Harper spectrum. Thus one deals with with energy levels if $Q = 1, 2, 3, 4, 5$. In contradistinction, competitions between gap and band structures are exhibited in the $x \equiv P$ dependence of related energies if $Q \geqslant 6$. It should also be mentioned that band structures can also be established in terms of non-unitary representations of pertinent spectrum-generating algebras, as shown recently for Scarf and Lamé Hamiltonians [35]. Further, we have succeeded in deriving closed formulae to the wavefunctions of the symmetrized q-difference form of the Harper equation. The q-normalization of such wavefunctions has been performed for the first time, now by applying the Jackson q-integral needed. The interesting point is the fact that the $I_{l_Q}^{(Q)}$ magnitudes of wavefunctions mentioned above lead to the appearance of peaks located around selected P-values for which the quotient P/Q is indivisible. The complementary P-values serve as separation points which are responsible for intermediary smooth bottomed valley-like structures, as shown in figure 4.

The conversion of the original Harper equation into the symmetrized q-difference equation is also worthy of note [36]. We emphasize, however, that for this purpose related but different

conversions proceeding in conjunction with admissible q-values can also be considered. It should also be mentioned that the band-energy spectrum of the Harper equation should be properly established by accounting for the k_i dependence of the Bloch wavefunction φ_B. This results in a more general second-order discrete equation like

$$\left(\frac{\exp(\mathrm{i}\theta_1)}{q^{n+\alpha_1+1/2}}+\frac{q^{n+\alpha_2+1/2}}{\exp(-\mathrm{i}\theta_2)}\right)\varphi(n+1)+\left(\frac{q^{n+\alpha_1-1/2}}{\exp(\mathrm{i}\theta_1)}+\frac{\exp(-\mathrm{i}\theta_2)}{q^{n+\alpha_2-1/2}}\right)\varphi(n-1)=E\varphi(n) \tag{41}$$

where $\varphi(n)=\varphi_B(na)$, $a=b$ and $\theta_i=k_ia$. This comes from the 'chiral' gauge proposed before [37], by now choosing the vector potential as

$$A_i=-B(x_1+x_2+\alpha_i a) \tag{42}$$

such that $x_i=n_ia$, $x_1+x_2=na$ and $n=n_1+n_2$, where n_i are integers. The q-difference counterpart of (41) is then easily obtained via $\psi(z)=\varphi(n)$ and $z=q^n$. Generalized BA equations concerning equations like (41) have already been proposed [38]. However, the derivation of explicit solutions is still an open problem which deserves further attention.

Acknowledgments

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